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論　文

A Numerical Study for Stability of Solutions in Latent Class Analysis by Ridge Approach

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ABSTRACT. This paper aims to show, through a numerical study, the fact that the latent parameters are extremely improved by using Gibson's method with a ridge parameter k. We will also propose a criterion for choosing an optimal value k from graphical considerations based on the relation between the ridge parameter k and a stability measure.

1. Introduction

Latent class anlysis aims to explain the response patterns of indivisuals by latent variables from data with the dichotomous responses. This analysis is widely used in social research. There are several methods which obtain the solutions from latent class analysis data. Methods by Green [1] and Gibson [2] are based on the structual equations, while MacHugh's [4] method are based on the maximum likelihood equation. However a stability of solutions in latent class analisys suffers a serious influence by small sampling errors, so frequently occur a phenomenon of improper solutions. Ikuzawa [3] has pointed out that Green's method often causes unreasonable solutions, and he has proposed a way to evade possibly the difficulty. Okamoto and Isogai [8] carried out a numerical study on three methods; Gibson's, Green's and Modified Green least squares method and found that the rounding of the manifest probabilities at the third decimal place affects the estimates of most parameters at the first decimal place.

In order to evade instability to which the latent class analysis inherently belongs and to get the reliability estimates, we have suggested some ideas [5], [6] and [7] so far. As a result of our study, in comparision between Gibson's and Green's method we showed a superiority of Gibson's method from numerical or theoretical points of view. And we have suggested a criterion on the choice of signatures and a stratifier of Gibosn's method.

In this paper we consider Gibson's method with a ridge parameter k and show how the Gibson's solutions are improved by varing k variously. Moreover from graphical considerations of parameter k and the stability measure, we shall give a criterion of choosing the ridge parameter k.

In Section 2 we state the outline of Gibson's method and in Section 3 propose the Gibson's method with the ridge parameter k. In Section 4 we shall show the results of numerical study.

2. Latent Class Model and Gibson's Method

Suppose there are n items and m classes and for each item the response of an individual is dichotomous and that an individual belongs to only one of m classes. Let us denote by $w_t (t = 1, \cdots, m)$ the probability that an individual belongs to class t and by $\pi_{it} (i = 1, \cdots, n)$ the probability that an individual belonging to class t responds positively to item i. We define the probabilities p_i, p_{ij} and p_{ijk}, which are called the manifest probabilities. The quantities p_i, p_{ij}, p_{ijk} are , respectively, the probabilities that an individual responds positively to item i, to both item i and j, and to item i, j and k simultaneously. Assuming local independence of the third-order, *i.e.*, the dependence of the response up to three items under the condition that an individual belongs to any latent class, we obtain

$$
\begin{aligned}
p_i &= \sum_{t=1}^{m} w_t \pi_{it}, \\
p_{ij} &= \sum_{t=1}^{m} w_t \pi_{it} \pi_{jt} \qquad (i \neq j), \\
p_{ijk} &= \sum_{t=1}^{m} w_t \pi_{it} \pi_{jt} \pi_{kt} \quad (i \neq j \neq k \neq i),
\end{aligned}
\tag{1}
$$

where $\sum_t^m w_t = 1$.

We call equation (1) the latent class model. Let $x_{id} = 1$ $(i = 1, \cdots, n; d = 1, \cdots, N)$ when an individual d responds positively to item i, and $x_{id} = 0$ otherwise, where N is the size of sample. Then we define the estimators of the manifest probabilities as follows:

$$
\begin{aligned}
\hat{p}_i &= \frac{1}{N} \sum_{d=1}^{N} x_{id}, \\
\hat{p}_{ij} &= \frac{1}{N} \sum_{d=1}^{N} x_{id} x_{jd}, \\
\hat{p}_{ijk} &= \frac{1}{N} \sum_{d=1}^{N} x_{id} x_{jd} x_{kd},
\end{aligned}
\tag{2}
$$

The purpose of latent class analysis is to estimate the set w_t, π_{it}, which are called the latent parameters, from the set $\hat{p}_i, \hat{p}_{ij}, \hat{p}_{ijk}$.

In this paper we discuss not the latent class model with $m \geq 3$ but $m = 2$. Now we will describe Gibson's method. For any fixed different two items, i and k, equation (1) imply the following expression:

$$T = L_1 D_k L_1^{-1}, \tag{3}$$

$$Q = L_1 D_w {}^t L_2, \tag{4}$$

where

$$T = (R\,{}^tQ)(Q\,{}^tQ)^{-1}, \quad R = \begin{pmatrix} p_k & p_{jk} \\ p_{ik} & p_{ij_sk} \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & p_{j_s} \\ p_i & p_{ij_s} \end{pmatrix},$$

$$L_1 = \begin{pmatrix} 1 & 1 \\ \pi_{i1} & \pi_{i2} \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 1 \\ \pi_{j_s1} & \pi_{j_s2} \end{pmatrix},$$

$$D_k = diag(\pi_{k1}, \pi_{k2}), \quad D_w = diag(w_1, w_2),$$

where $j_s, s = 1, \cdots, n-2 (j_s \neq k, i), T, L_1, D_k$, and D_w are matrices of type 2×2, and R, Q and ${}^tL_2$ are of type $2 \times (n-1)$. Item i and k are called the left signature and the stratifier, respectively, and the other items $j_s^{\prime}s$ are called the right signature. We estimate R and Q from (2) and solve the eigenvalues problem with T of (3), so that we obtain D_k and L_1 as the eigenvalues and eigenvectors, respectively. From equation (4) we can also estimate D_w and L_2 by using the estimate Q and L_1 obtained above.

3. Ridge Approach to Gibson's Method

When solving the eigenvalues problem with T by Gibson's method, the eigenvalues and eigenvectors of T often become unstable because the matrix $Q\,{}^tQ$ trends to a semi-singular matrix (see [8]). To avoid this difficulty, we introduce a ridge parameter k which is often used in regression analysis, and we will attempt stabilizing the solutions. A matrix T in Gibson's method with the ridge parameter k is defined as follows:

$$T = (R^tQ)(Q^tQ + kI)^{-1}.$$

We inspect an optimal value of k by varing k according with the choice of a stratifer and signatures. However we, in general, do not have the criterion on the choice of k, as well as regression analysis. In this paper we adopt a criterion ε which are suggsted in [7]. And we use EWP as a stability measure:

$$\varepsilon = (trace(T)^2 - 4det(T))^{\frac{1}{2}},$$

$$EWP = \Big\{\frac{1}{2(n+1)}(\sum_{t=1}^{2}\Delta w_t^2 + \sum_{i=1}^{n}\sum_{t=1}^{2}\Delta\pi_{it}^2)\Big\},$$

where t_{12} is the element in first row and second column of T, $\Delta\pi_t = \hat{w}_t - w_t$ and $\Delta\pi_{it} = \hat{\pi}_{it} - \pi_{it}$.

4. Results of Numerical Study

A latent class model is shown in Table 1. We shall study a numerical model with $m = 2$ and $n = 4$ described in Table 2. We utilized the data by rounding-off the manifest probabilities, which are generated from Table 2, at the third decimal place (see [8]).

Table 1. Latent Class Model

Class		Item 1	2	...	...	...	n
1	w_1	π_{11}	π_{21}	...	...	...	π_{n1}
2	w_2	π_{12}	π_{22}	...	...	...	π_{n2}
⋮				⋮			
m	w_m	π_{1m}	π_{2m}	...	...	...	π_{nm}

Table 2. Numerical Model

Class		Item 1	2	3	4
1	.6	.6	.7	.7	.5
2	.4	.4	.3	.2	.4

In Fig.1 to Fig.12, we show the graphs when varing k. The vertical axis of the upper graph and the lower graph of each of Figure, respectively, indicates anEWP value and a criterion ε value. Table 3 shows estimates of latent parameters which are obtained by solving in the eigenvalues problem of T without k and Table.4 with k, where k is the point at which an approximate minumum value of EWP is attained.

Conclusion

1. The situation of the change of EWP and ε corresponds well each other as seen from Fig.1 to Fig.12. Actually the value EWP is unknown and therefore we can not get the graph of EWP on the parameter k, the other the criterion ε is a function in the only k as given data, thus we can draw the graph and obtain the approximate value k which minimize ε. Using this k, we solve the eigenvalues problem of T.

2. From Table 3 and 4 it is seen that by using a suitable parameter k, the estimates of latent parameter are drastically improved. For example Case 3 and Case 10 have improper solutions on the usual Gibson's method, but our proposal method introduced the ridge parameter k gives outstanding solutions.

In the forthcoming paper we will report the theoretical validity on Gibson's method having the ridge parameter k and on the criterion ε.

References

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Table 3. Estimates by an usual Gibson method

No.	w_1	π_{11}	π_{21}	π_{31}	π_{41}	π_{12}	π_{22}	π_{32}	π_{42}	EWP
1	.32	.65	.87	.82	.53	.46	.38	.35	.43	.15
2	.33	.65	.84	.84	.52	.46	.39	.33	.43	.15
3	.01	1.03	7.84	.680	.75	.52	.50	.47	.46	2.99
4	.38	.64	.84	.78	.47	.45	.37	.33	.45	.12
5	.33	.65	.86	.82	.52	.46	.38	.34	.43	.15
6	.42	.54	.85	.72	.52	.51	.33	.34	.41	.11
7	.38	.64	.80	.81	.47	.45	.38	.32	.45	.12
8	.33	.64	.86	.82	.53	.46	.38	.34	.42	.15
9	.44	.54	.78	.76	.52	.51	.35	.31	.41	.10
10	.01	1.04	7.34	7.58	.75	.52	.50	.46	.46	3.04
11	.27	.64	.96	.81	.54	.47	.38	.38	.43	.19
12	.27	.67	.90	.87	.54	.47	.41	.36	.43	.18

Table 4. Estimates by new proposal method

No.	w_1	π_{11}	π_{21}	π_{31}	π_{41}	π_{12}	π_{22}	π_{32}	π_{42}	EWP	$k(\times 10^{-3})$
1	.60	.59	.74	.66	.50	.41	.23	.26	.40	.033	.35
2	.58	.60	.70	.72	.49	.41	.31	.19	.41	.011	.31
3	.69	.57	.65	.59	.51	.43	.30	.30	.35	.066	.10
4	.59	.61	.72	.65	.47	.39	.31	.29	.45	.058	.15
5	.60	.58	.70	.72	.49	.43	.30	.17	.41	.017	.15
6	.60	.53	.74	.63	.51	.50	.28	.31	.38	.059	.025
7	.59	.61	.68	.68	.47	.39	.34	.26	.45	.032	.10
8	.59	.58	.76	.66	.50	.43	.21	.27	.41	.044	.15
9	.58	.53	.70	.68	.51	.50	.32	.27	.39	.047	.025
10	.54	.62	.71	.67	.50	.40	.34	.30	.41	.044	.25
11	.59	.58	.77	.65	.50	.44	.21	.29	.40	.050	.25
12	.60	.59	.71	.70	.59	.42	.29	.20	.42	.009	.27

(1998年12月1日　受理)

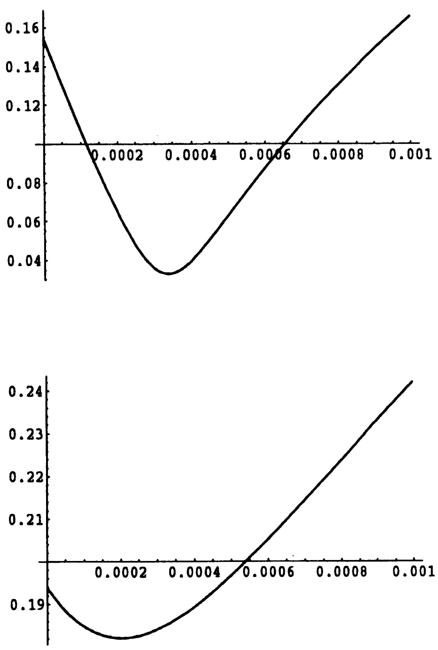

Fig.1. case 1

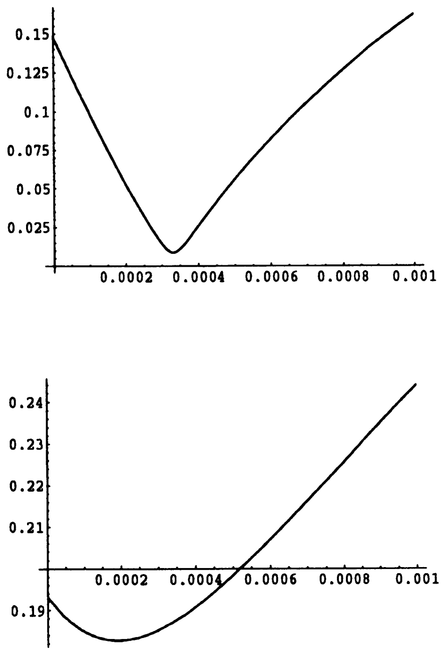

Fig.2. case 2

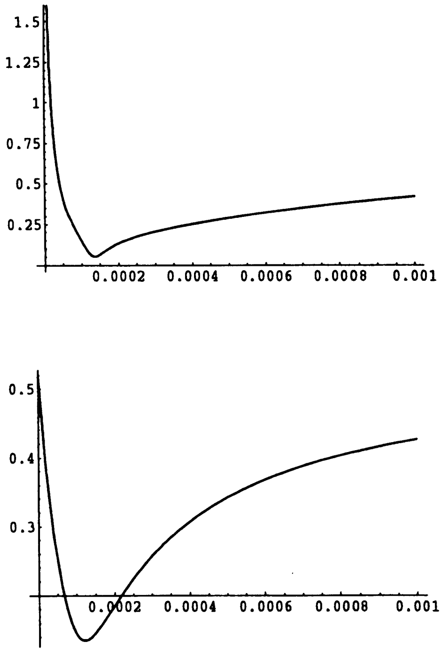

Fig.3. case 3

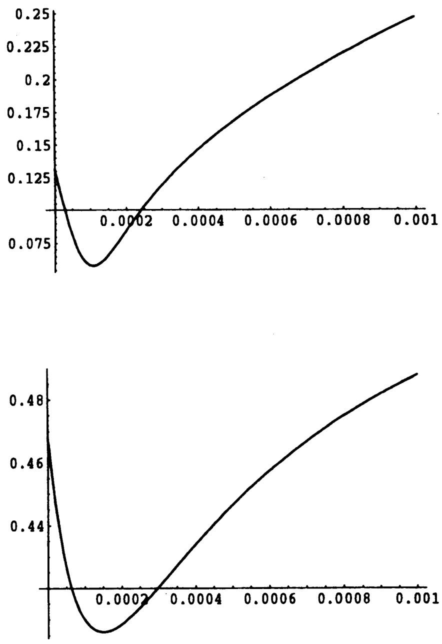

Fig.4. case 4

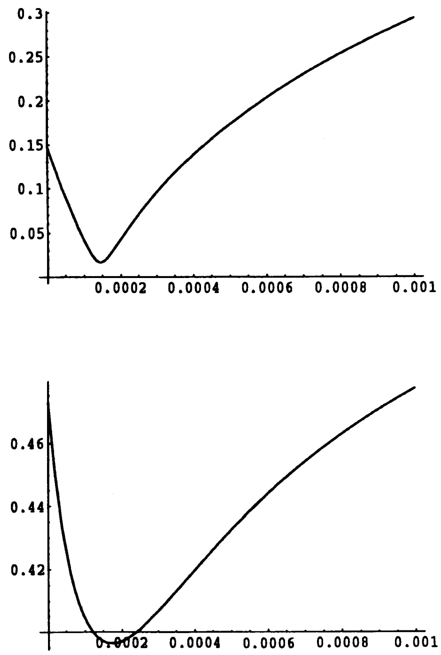

Fig.5. case 5

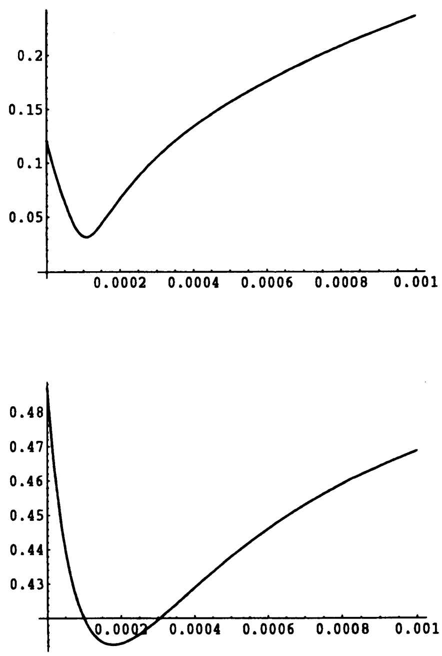

Fig.7. case 7

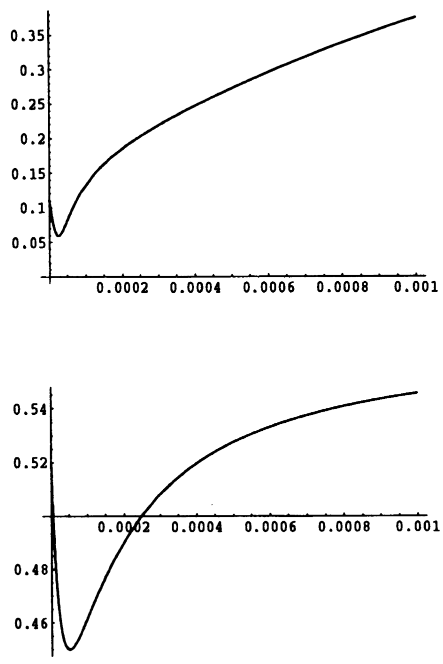

Fig.6. case 6

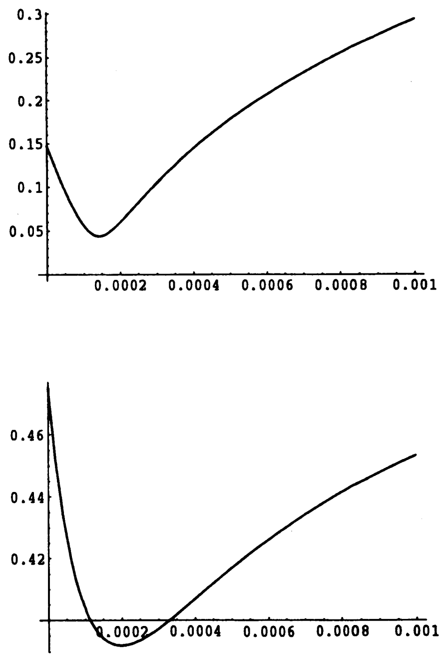

Fig.8. case 8

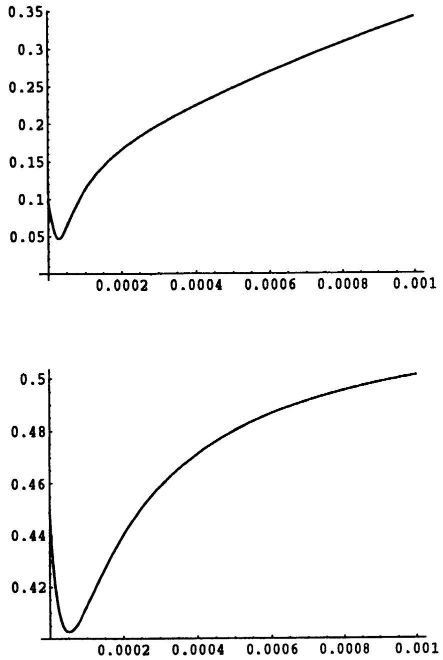

Fig.9. case 9

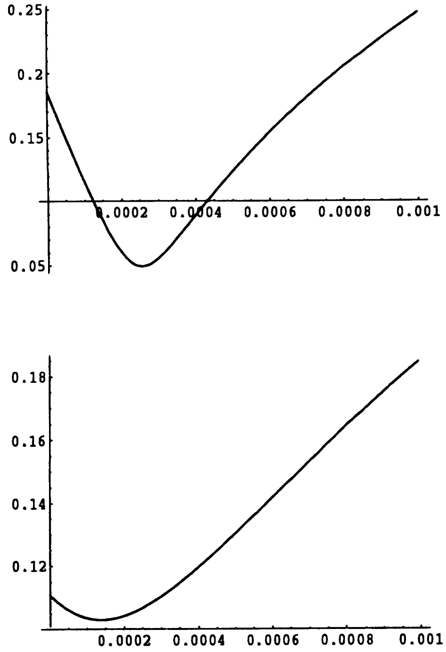

Fig.11. case 11

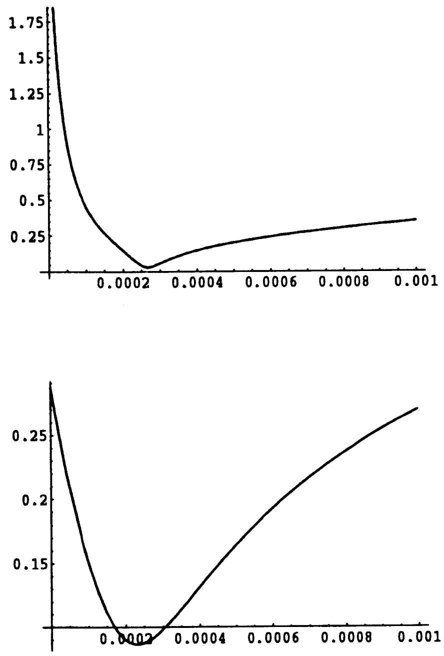

Fig.10. case 10

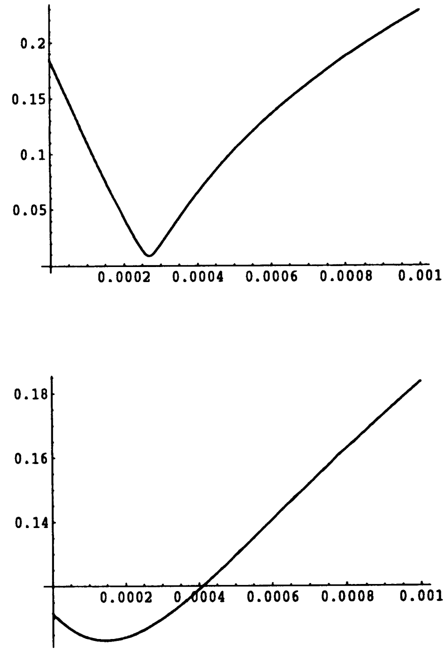

Fig.12. case 12